NONADDITIVE INFORMATION THEORY FOR THE ANALYSIS OF BRAIN RHYTHMS

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Abstract-**In this paper, we introduce Nonadditive Information Theory through the axiomatic formulation of Tsallis entropy. We show that systems with transitions from high dimensionality to few degrees of freedom are better described by nonadditive formalism. Such a biological system is the brain and brain rhythms is its macroscopic dynamic trace. We will show with simulations that Tsallis entropy is a powerful information measure, and we present results of brain dynamics analyzed using EEG recordings from a brain injury model.**

Keywords-**Tsallis Entropy, EEG, Ischemia**

I. INTRODUCTION

Nonadditivity, or nonextensivity, is an important concept in the field of information theory. A system is nonextensive if it contains long-range interaction, long-range memory, or (multi)fractal structure [1],[2],[3]. In such a system, a macroscopic dynamic quantity is not simply proportional to the microscopic degrees of freedom but there are moments where the system unpredictably looses its high dimensionality (described by stochastic process) and falls to a low dimensional deterministic system (either linear or not). Classical paradigms in brain dynamics is the rise of an epileptic crisis or the burst suppression EEG following the resuscitation from cardiac arrest or asphyxia [13],[14]. In this respect, a nonextensive generalization of Boltzmann-Gibbs statistical mechanics formulated by Tsallis [4] is better suited to describe such phenomena.

II. NONADDITIVE INFORMATION THEORY

In the development of the foundations of classical information theory, Khinchin [5] presented a mathematically rigorous proof of a uniqueness theorem for the Shannon entropy based on the additivity law for a composite system in terms of the concept of conditional entropy. Suppose the total system can be divided into two subsystems, *A* and *B*, and let $p_{ij}(A,B)$ be the normalized joint probability of finding *A* and *B* in their *i*th and *j*th states, respectively. Then the conditional probability of *B* given *A* found in its *i*th state is given by $p_{ij}(B|A) = p_{ij}(A,B) / p_i(A)$, which leads to the celebrated Bayes multiplication law

$$p_{ij}(A,B) = p_i(A)\, p_{ij}(B|A), \tag{1}$$

where $p_i(A)$ is the marginal probability distribution: $p_i(A) = \sum_j p_{ij}(A,B)$. It should be noted that this form of factorization can always be established in any physical situation. The Shannon entropy of the composite system is

$$S(A,B) = -k_B \sum_{i,j} p_{ij}(A,B) \ln p_{ij}(A,B)^{1}. \tag{2}$$

Combining (1) and (2) yields

$$S(A,B) = S(A) + S(B|A), \tag{3}$$

where $S(B|A)$ stands for the conditional entropy [6].

In the particular case when *A* and *B* are statistically independent, $p_{ij}(B|A)=p_j(B)$ and from (1) and (2) is drawn out the additivity law $S(A, B) = S(A) + S(B)$. We emphasize here that there is a natural correspondence relation between the multiplication law and the additivity law:

$$p_{ij}(A,B) = p_i(A) p_{ij}(B|A) \Leftrightarrow S(A,B) = S(A) + S(B|A), \tag{4}$$

When the above discussion is generalized to any composite system there are theoretical and experimental considerations where systems do not obey to the additivity law [7]. In this respect, a nonextensive generalization of Boltzmann-Gibbs statistical mechanics formulated by Tsallis [4] is better suited to describe such phenomena. In this formalism, referred to as nonextensive statistical mechanics, Shannon entropy in (2) is generalized as follows:

$$S_q(A,B) = \frac{1}{1-q}\left\{\sum_{i,j} \left[p_{ij}(A,B)\right]^q - 1\right\}, \tag{5}$$

where q is a positive parameter. This quantity converges to the Shannon entropy in the limit $q \to 1$. Like the Shannon entropy, it is nonnegative, possesses the definite concavity for all q>0, and is known to satisfy the generalized H-theorem . Nonextensive statistical mechanics has found a lot of physical applications. A standard discussion about the nonadditivity of the Tsallis entropy $S_q(p)$ assumes factorization of the joint probability distribution in (1) ($p_{ij}(A,B) = p_i(A) p_j(B)$). Then, the Tsallis entropy is found to yield the so-called pseudoadditivity

$$S_q(A,B) = S_q(A) + S_q(B) + (1-q) S_q(A) S_q(B). \tag{6}$$

Clearly, the additivity holds if and only if $q \to 1$. However, there is a logical difficulty in this discussion. As mentioned

[1] We use throughout the paper dimensionless units where the Boltzmann constant, k_B is set equal to 1

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above, Tsallis' nonextensivity was devised in order to treat a system with, for example, long-range interactions. On the other hand, physically, "dividing the total system into the subsystems" implies that the subsystems are made spatially separated in such a way that there is no residual interaction or correlation. If the system is governed by a long-range interaction, the statistical independence can never be realized by any spatial separation since the influence of the interaction persists at all distances. In fact, the probability distribution in nonextensive statistical mechanics does not have a factorizable form even if systems A and B are dynamically independent, and therefore correlation is always induced by nonadditivity of statistics [8].

Thus, it is clear that the assumption of the factorized joint probability distribution is not physically pertinent for characterizing the nonadditivity of the Tsallis entropy. These considerations naturally lead us to the necessity of defining the conditional entropy associated with the Tsallis entropy.

To overcome the above mentioned logical difficulty and to generalize the correspondence relation in (4) simultaneously, Santos [9] proposed a generalization of Shannon's theorem to Tsallis entropy, Hotta et Joichi [10] investigated composability and generalized (Tsallis) entropy and Abe [11] extended the work of Santos [9] generalizing the Khinchin axioms for the ordinary information theory in a natural way to the nonextensive systems.

Along the lines of recent paper of Abe and Rajagopal [12] we consider the Tsallis entropy of the conditional probability distribution $p_{ij}(A|B) = p_{ij}(A,B)/p_i(A)$ as

$$S_q(B|A) = \frac{S_q(A,B) - S_q(A)}{1 + (1-q)S_q(A)}. \quad (7)$$

From this, we immediately see that

$$S_q(A,B) = S_q(A) + S_q(B|A) + (1-q)S_q(A)S_q(B|A) \quad (8)$$

which is a natural nonadditive generalization of (3) in view of pseudoadditivity in (6). Therefore the correspondence relation in (4) becomes now

$$p_{ij}(A,B) = p_i(A)\,p_{ij}(B|A) \Leftrightarrow$$

$$S_q(A,B) = S_q(A) + S_q(B|A) + (1-q)S_q(A)S_q(B|A). \quad (9)$$

This equation coincides with the (6) of pseudoadditivity when the two systems *A* and *B* are independent.

In this way the nonadditive Tsallis entropy was formulated according to the Khinchin axioms of information theory and the contradiction between dependency (Bayes law) and long range interaction (nonadditivity) is removed. Summarizing, we have established on firm mathematical grounds, a general criterion for consistent testing of the independence between random variables, which we propose as a practical tool to analyze the EEG recordings. The results depends upon the entropic index *q*. It is expected that, for every specific use, better discrimination can be achieved with appropriate ranges of values *q*.

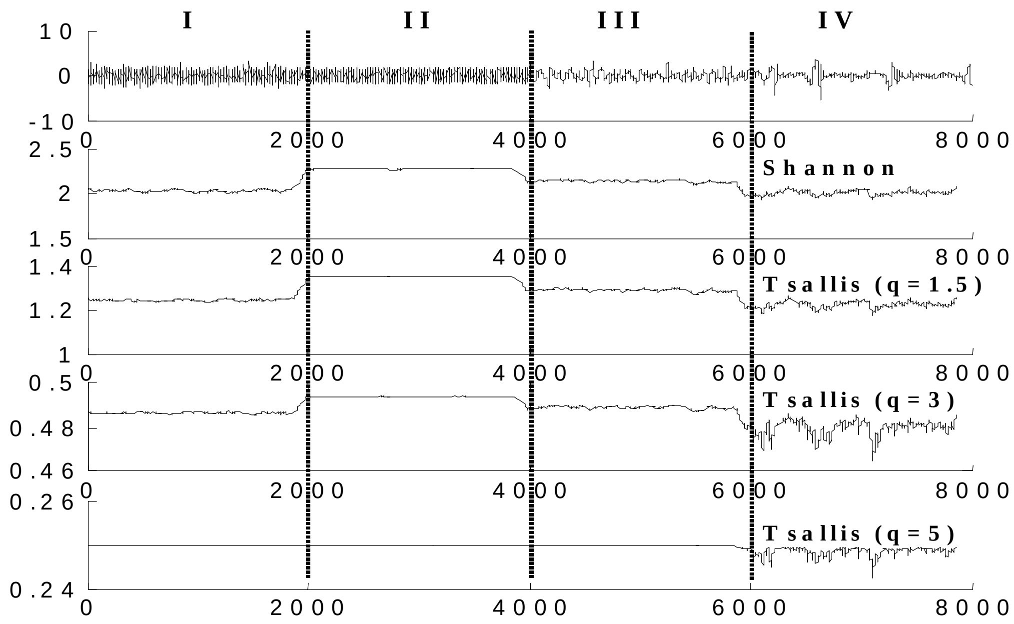

Fig. 1. A synthetic signal (top) and Shannon and Tsallis entropy for different *q* index values(q=1.5,3,5). The synthetic signal is composed by: (I) one realization of Gaussian distribution with zero mean and standard deviation equals 1, (II) one realization of uniform distribution in the interval [-2,2], (III) EEG recordings from baseline (III) and early recovery (IV). For q=3 the Tsallis entropy is sensitive enough to distinguish between signals with different probability distribution function and to differentiate the baseline EEG from early recovery EEG.

III. SIMULATIONS

We explore the use of parameterized Tsallis entropy (function of *q*) to monitor the changes of EEG undergoes in different conditions and to compare it with the entropy of signals with known probability distribution function (PDF). In particular, we compare the Tsallis and Shannon entropy measures on a synthetic signal. The synthetic signal *s(k)* is composed of four parts of 2000 samples each with different PDF's: (I) Gaussian distribution with zero mean and standard deviation $\sigma = 1$, (II) a uniform distribution in the interval [-2,2], (III) an EEG recording from baseline, and (IV) an EEG recording from early recovery after ischemic/cardiac arrest brain injury [13]. Figure 1, from top to bottom, shows one realization of signal *s(t)* and Shannon and Tsallis entropies (q=1.5, 3 and 5) after 50 runs.

The following points are of importance:

1) The entropy is maximum for uniform distribution. The entropy of baseline EEG lies between the two extreme distributions; Gaussian and uniform, preserving a subgaussian distribution.
2) Transition from one distribution to the next is properly revealed.
3) Shannon and Tsallis entropy for q=1.5 are more sensitive than Tsallis entropy for q=3 and 5 for segmenting the different parts of the signal.
4) Tsallis entropy for q=5 has greater discriminant power between parts I,II,III and the part of the signal from recovery EEG (IV).

IV. CONCLUSIONS

We have constructed the nonadditive conditional entropy based on the axiomatic foundations of classical information theory and the pseudoadditivity of the Tsallis entropy. Furthermore we have shown examples where the nonadditive entropy provides a novel statistical description of the brain rhythms during asphyxic injury and recovery.

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